

Weak Double Layers: Existence, Stability, Evidence

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Two more classes of monotonic double layers complementing the class of beam-type double layers are investigated analytically, and their range of existence is explored in the small amplitude limit. One class preferentially exists for hot ions and electron drifts of the order of electron thermal velocity. The second one, instead, assumes hot electrons and needs almost current-free conditions. The first class, called SEADL, is based on the slow electron acoustic branch and exhibits a tuning-fork configuration in the electron phase space. Its density decreases with increasing potential. The second one (SIADL) rests on the slow ion acoustic branch and, consequently, has a tuning-fork pattern in the ion phase space. Its density increases with the potential. Both classes are found to be linearly stable with respect to one-dimensional, but unstable with respect to two-dimensional electrostatic perturbations. A comparison with experiments suggests an identification of the second type with the double layers obtained by Hollenstein.

1. Introduction and Summary

The implications and physical relevance of the existence of electrostatic double layers (DLs) are twofold: these field configurations are able to accelerate certain groups of particles and simultaneously to confine other groups of particles as well. Whereas it is mainly the first concept being used in astrophysical, terrestrial or laboratory plasmas [1–6], it is the electrostatic confinement aspect that has received interest in fusion research [7]; for example, in tandem mirror machines the maintainability of potential barrier against end-loss is one of the critical issues.

To prevent misunderstanding, some words should be said at the beginning to the definition of DLs used in this paper. Here, a DL is understood by its original definition, namely as a monotonic transition of the electrostatic potential which connects two differently biased asymptotic plasmas. Such a configuration is consistent with a localized dipole-sheet of space charge surrounded by a field-free region. More general definitions include superimposed holes and humps, but, as it has been shown in [8–11], these new classes of nonlinear wave solutions can be completely understood in terms of asymmetric phase space vortices (holes), and one should, therefore, insist on its original, certainly more correct definition. We are, hence, not adopting a preliminary classification of DLs introduced by

the author [8, 9], and mean, further on, by a DL a strictly monotonic transition. The general class of DLs, therefore, consists of the former class of type-I-DLs which has to be extended by two further branches, as we will see. The type-II-DL and the type-III-DL in the old nomenclature are, hence, denoted by asymmetric ion holes and by asymmetric electron holes, respectively, as it has been used already in [8, 9] in conjunction with the other designations.

The known solution of strong DLs [8, 9, 12] which satisfies the requirement of physical distributions (*i.e.* with respect to smoothness and asymptotic behaviour) can be characterized as follows:

- in addition to the free streaming particles there must be a well separated beam-type trapped particle distribution in each component,
- the free particle distributions have to satisfy Bohm-like criteria, *i.e.* the free particles have to enter the DL region with a finite velocity,
- there does not exist a small amplitude limit,
- the asymptotic densities are more or less equal.

To have a short-hand notation we'll refer to beam-like DL (BDL) or, as one wishes, to Bohm-like DL.

In addition, there are different, sometimes non-physical solutions found in literature of monotonic transitions [11, 13–16]. It was, however, only very recently that a further progress could be achieved. Based upon the formalism developed by the author [17], KIM [18] established the existence of two more classes of monotonic DLs and presented the first

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analytic expression for a realistic electrostatic DL potential. These new solutions do have a small amplitude limit and are related to the hole solutions in a similar but not identical way as the aforementioned ones.

The present paper reinforces this point of view by generalizing the theory and by exploiting the circumstances under which these new solutions exist. Moreover, a stability analysis is performed and a first comparison with experiments is made.

The main results are summarized in Fig. 1 where the three types of DLs are plotted schematically. The first column refers to the BDL solution and is shown for comparison. From top to bottom, the potential, the ion phase space, the electron phase space, and the various densities are drawn. We clearly can distinguish the aforementioned properties of this kind of DL: the beam type character of the distributions, the finite drift velocities u_0 and v_0 , and the asymptotic equality of the densities.

The second column shows the corresponding quantities of the so-called slow electron acoustic DL (SEADL). First, we realize a huge change in the phase space configurations. The separation between free and trapped particles is no longer visible, as the separatrix, represented by the dashed line, is completely dived into the distribution. The ion drift is essentially reduced, and the electron drift has become smaller too. As the analysis shows, the electron drift is basically introduced by the relationship to the slow electron acoustic mode [17, 19]. It's, therefore, reasonable to term this kind of DL slow electron acoustic DL. Whereas the trapped ion region is filled up completely, the electron's counterpart is almost empty. The few trapped electrons are located near the separatrix, *i.e.* near the crest of the wave. The electron phase space pattern resembles a tuning-fork. Consequently, the total density diminishes with increasing Φ , the asymmetry being produced by the large amount of

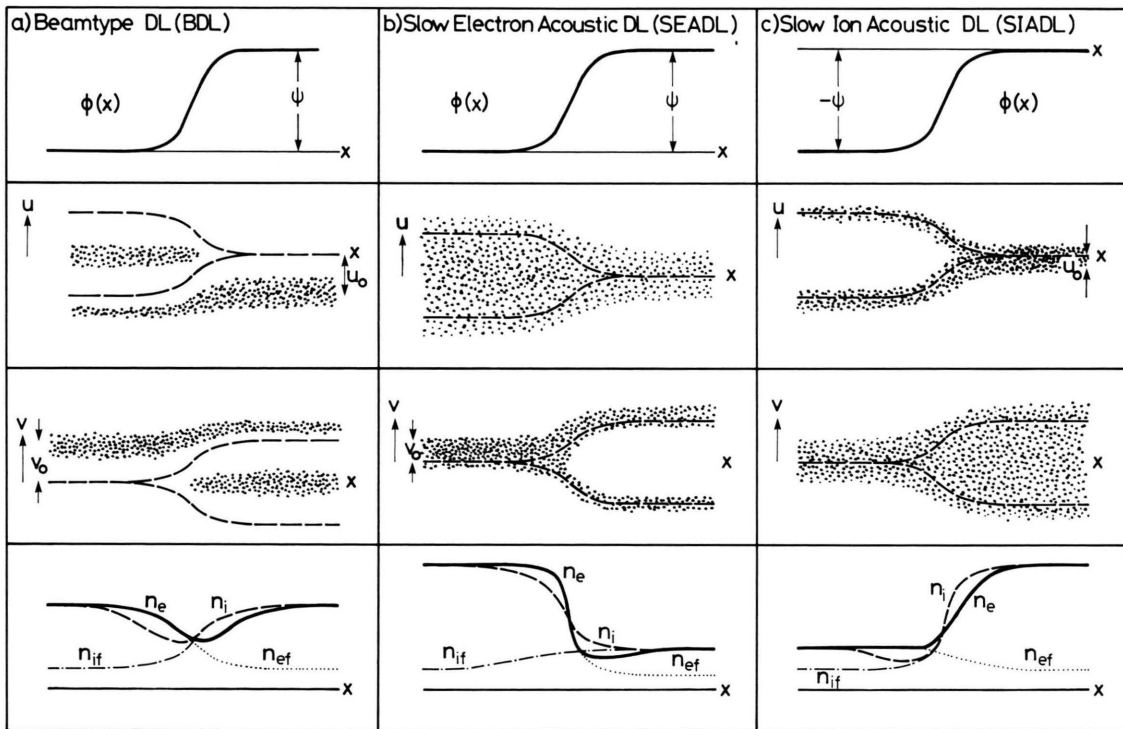


Fig. 1. A schematic plot of the three types of double layers; the first, second, and third column refer to the beam-type, the slow electron acoustic type, and the slow ion acoustic type double layer, respectively. The first row shows the electrostatic potential, the second and third one the corresponding ion and electron phase space, respectively. The separatrices are indicated by dashed lines. In the last row the various densities are drawn as functions of x , the total electron and ion densities, n_e (solid curve), and n_i (broken curve), respectively, and the densities of free electrons n_{ef} (dotted curve), and of free ions n_{if} (dashed-dotted curve).

trapped ions on the low potential, and by a deficiency of trapped electrons on the high potential side. This is also manifest in the density plot where the solid (dashed) line represents the total electron (ion) density; the dotted (dotted-dashed) line refers to the free electron (ion) contribution. Furthermore, the ion temperature, defined at $x \rightarrow \infty$, is usually higher than the electron temperature which is defined at $x \rightarrow -\infty$.

The third column shows the slow ion acoustic DL (SIADL) and its related quantities. It is just the opposite of the previous solution. Electrons and ions are simply interchanged if one switches from the slow electron to the slow ion acoustic DL. The tuning-fork configuration does now appear in the ion phase space.

A finite plasma current turns out to be necessary for SEADL.

More details, including the stability behaviour against localized modes, and the experimental evidence are found in the corresponding sections.

We now turn to an explicit evaluation of the analysis.

The Vlasov theory of one-dimensional electrostatic equilibria has been reviewed and developed in [8] and the papers cited therein. We, therefore, proceed directly to the construction of the corresponding equilibrium solutions. In both cases the analysis is performed in the small amplitude limit. Throughout the paper normalized quantities are used. The electron (ion) quantities are normalized by the free electron (ion) quantities at $x \rightarrow -\infty$ ($x \rightarrow +\infty$), e.g., the electron velocity is normalized by the electron thermal velocity at a region where the trapped electrons are absent.

We first draw our attention to the slow electron acoustic DL (SEADL).

2. The Slow Electron Acoustic Double Layer (SEADL)

The stationary Vlasov equations for electrons and ions, respectively, are solved by the following ansatz:

$$\begin{aligned} f_e(x, v) &= (2\pi)^{-1/2} \begin{cases} \exp\{-\frac{1}{2}[\sigma_e(v^2 - 2\Phi)^{1/2} - v_0]^2\}, & \varepsilon_e > 0, \\ \exp(-v_0^2/2) \exp[-\beta(v^2 - 2\Phi)/2], & \varepsilon_e \leq 0, \end{cases} \\ f_i(x, u) &= \tilde{A}(2\pi)^{-1/2} \begin{cases} \exp\{-\frac{1}{2}[\sigma_i(u^2 - 2\theta(\psi - \Phi))^{1/2} - u_0]^2\}, & \varepsilon_i > 0, \\ \exp(-u_0^2/2) \exp(-\alpha[u^2 - 2\theta(\psi - \Phi)]/2), & \varepsilon_i \leq 0, \end{cases} \end{aligned} \quad (1)$$

where the following short-hand notation has been used

$$\begin{aligned} \sigma_e &= \text{sgn } v, & \sigma_i &= \text{sgn } u, \\ \varepsilon_e &= (v^2 - 2\Phi)/2, & \varepsilon_i &= [u^2 - 2\theta(\psi - \Phi)]/2, \\ \theta &= T_e/T_i. \end{aligned} \quad (2)$$

v_0 and u_0 , respectively, are the drift velocities of the streaming (free) particles in the region where the trapped particles are absent. The trapping parameters α and β control the density of trapped particles which are represented by the second inequalities in (1). Both signs are admitted for α and β , the negative sign corresponding to a hole in the distribution function. The potential drop is given by the normalized quantity ψ . \tilde{A} is a normalization constant. Both distributions are continuous, a necessary requisite for a physically acceptable theory. They are, furthermore, derivatives of Maxwellians and reduce to them in appropriate limits.

Performing the velocity integration we get [17]

$$\begin{aligned} n_e(\Phi) &= \exp(-v_0^2/2) [F(v_0^2/2, \Phi) + T_{\pm}(\beta, \Phi)], \\ n_i(\Phi) &= \tilde{A} \exp(-u_0^2/2) [F(u_0^2/2, \theta(\psi - \Phi)) \\ &\quad + T_{\pm}(\alpha, \theta(\psi - \Phi))], \end{aligned} \quad (3)$$

where F and T_{\pm} stand for the contributions of free and trapped particles, respectively, and \pm refers to the sign of the involved trapping parameter.

They can be written as

$$F(v_0^2/2, \Phi) = \exp \Phi \text{erfc}(\Phi^{1/2}) + K(v_0^2/2, \Phi), \quad (4)$$

$$T_{\pm}(\beta, \Phi) = \begin{cases} \beta^{-1/2} \exp(\beta\Phi) \text{erf}[(\beta\Phi)^{1/2}], & \beta > 0, \\ 2(\pi|\beta|)^{-1/2} W[(-\beta\Phi)^{1/2}], & \beta \leq 0, \end{cases}$$

where $K(x, y)$ can be found in [8, 9, 17], and $W(x)$ is Dawson's function. We need the small amplitude

expansion of (3) and (4) which becomes

$$F(v_0^2/2, \Phi) + T_{\pm}(\beta, \Phi) \\ = \exp(v_0^2/2) \left\{ 1 - \frac{1}{2} Z'_r(v_0/\sqrt{2}) \Phi - \frac{4}{3} b(\beta, v_0) \Phi^{3/2} + \frac{1}{2} G(v_0) \Phi^2 + \dots \right\}, \quad 0 \leq \Phi \leq 1. \quad (5)$$

Note that (5) is valid for both signs of β . In (5) $Z'_r(x)$ is the real part of the Fried-Conte-dispersion function, and $b(\beta, v_0)$ is defined by

$$b(\beta, v_0) \equiv \pi^{-1/2} (1 - \beta - v_0^2) \exp(-v_0^2/2), \quad (6)$$

and $G(v_0)$ is a monotonic decreasing function of v_0 with $G(0) = 1$, and

$$G(v_0) = \exp(-v_0^2/2) + 3v_0^{-4}, \quad |v_0| \gg 1.$$

Applying (5) to (3) and using the charge neutrality condition at infinity, $n_e = n_i = 1$ for $\Phi = 0$, we get

$$n_e(\Phi) = 1 - \frac{1}{2} Z'_r(v_0/\sqrt{2}) \Phi - \frac{4}{3} b(\beta, v_0) \Phi^{3/2} \\ + \frac{1}{2} G(v_0) \Phi^2 + \dots, \quad (7)$$

$$n_i(\Phi) = 1 + \frac{\theta}{2} Z'_r(u_0/\sqrt{2}) \Phi + \frac{4}{3} b(\alpha, u_0) \\ \cdot \theta^{3/2} [\psi^{3/2} - (\psi - \Phi)^{3/2}] + \frac{\theta^2}{2} G(u_0) \Phi^2 \\ - \theta^2 \{ G(u_0) - [\frac{1}{2} Z'_r(u_0/\sqrt{2})]^2 \} \Phi \psi + \dots$$

valid for $0 \leq \Phi \leq \psi \ll 1$, and $\theta \psi \ll 1$. Apparently, the microscopic details enter in (7) through the coefficients of the half power expansion in Φ .

Substituting this expression into Poisson's equation

$$\Phi'' = n_e - n_i, \quad (8)$$

we obtain

$$\Phi'' = A \Phi + B_1 \Phi^{3/2} + B_2 [\psi^{3/2} - (\psi - \Phi)^{3/2}] \\ + C_1 \Phi^2 + C_2 \Phi \psi + \dots \equiv - \frac{\partial V}{\partial \Phi}, \quad (9)$$

where

$$A = -\frac{1}{2} [\theta Z'_r(u_0/\sqrt{2}) + Z'_r(v_0/\sqrt{2})], \\ B_1 = -\frac{4}{3} b(\beta, v_0), \\ B_2 = -\frac{4}{3} b(\alpha, u_0) \theta^{3/2}, \\ C_1 = \frac{1}{2} [G(v_0) - \theta^2 G(u_0)], \\ C_2 = \theta^2 \{ G(u_0) - [\frac{1}{2} Z'_r(u_0/\sqrt{2})]^2 \}. \quad (10)$$

A first relation among these five constants A , B_1 , B_2 , C_1 , and C_2 is obtained by applying the second charge neutrality condition, $n_e = n_i$ at $\Phi = \psi$, which

specifically holds for monotonic DLs:

$$0 = A + (B_1 + B_2) \psi^{1/2} + (C_1 + C_2) \psi. \quad (11)$$

Multiplication of (9) by Φ' and integration yields the "energy law"

$$\frac{\Phi'(x)^2}{2} + V(\Phi) = 0 \quad (12)$$

with the "classical potential" $V(\Phi)$ given by

$$-V(\Phi) = \frac{A}{2} \Phi^2 + \frac{2}{5} B_1 \Phi^{5/2} \\ + B_2 \{ \Phi \psi^{3/2} - \frac{2}{5} [\psi^{5/2} - (\psi - \Phi)^{5/2}] \} \\ + \frac{C_1}{3} \Phi^3 + \frac{C_2}{2} \Phi^2 \psi + \dots \quad (13)$$

In (13) $V(0) = 0$ has been assumed.

Vanishing electric field $E = -\Phi'$ at infinity implies $V(\psi) = 0$ which reads

$$0 = A + \frac{2}{5} [2B_1 + 3B_2] \psi^{1/2} + \left[\frac{2C_1}{3} + C_2 \right] \psi. \quad (14)$$

(14) is usually called the nonlinear dispersion relation as it determines the phase velocity of the nonlinear wave in terms of its amplitude. For electron holes where (11) is not assumed, (14) is the only decisive condition. The DL solution we are looking for has, in addition, to satisfy (11) and is, therefore, a specialization of the former.

Solving (11) and (14) in terms of B_2 , C_1 , and C_2 , we get

$$B_1 = B_2 - \frac{5}{3} C_1 \psi^{1/2}, \\ A = -2B_2 \psi^{1/2} + \frac{2}{3} C_1 \psi - C_2 \psi, \quad (15)$$

which are used to reduce the classical potential:

$$-V(\Phi) = B_2 \{ \psi^{1/2} \Phi (\psi - \Phi) \\ - \frac{2}{5} [\psi^{5/2} - \Phi^{5/2} - (\psi - \Phi)^{5/2}] \} \\ + \frac{C_1}{3} \Phi^2 [\psi^{1/2} - \Phi^{1/2}]^2. \quad (16)$$

It is worth to mention that $V(\Phi)$ does not depend on the parameter C_2 .

For convenience, we also write down the first and second derivative of $V(\Phi)$

$$-V'(\Phi) = B_2 \{ \psi^{3/2} - 2\psi^{1/2}\Phi + \Phi^{3/2} - (\psi - \Phi)^{3/2} \} + \frac{C_1}{3} \Phi \{ 2\psi - 5(\Phi\psi)^{1/2} + 3\Phi \}, \quad (17)$$

$$-V''(\Phi) = B_2 \{ -2\psi^{1/2} + \frac{3}{2}\Phi^{1/2} + \frac{3}{2}(\psi - \Phi)^{1/2} \} + \frac{C_1}{3} \{ 2\psi - \frac{15}{2}(\Phi\psi)^{1/2} + 6\Phi \}. \quad (18)$$

This potential uniquely defines a double layer, provided that the two equations in (15) and the requirement $V < 0$ for $0 < \Phi < \psi$, can be satisfied.

To show this, we first investigate the second equation in (15). With $\psi \ll 1$ the rhs of (15) is small and A has to be nearly zero

$$-\frac{1}{2} [\theta Z_r'(u_0/\sqrt{2}) + Z_r'(v_0/\sqrt{2})] \approx 0. \quad (19)$$

The function $-\frac{1}{2} Z_r'(x)$, plotted in Fig. 2, is unity for $x = 0$, vanishes for $x = 0.924$, and is negative for larger values of x . More precisely, it holds

$$-\frac{1}{2} Z_r'(x) = \begin{cases} -\frac{(x - x_0)}{x_0} + (x - x_0)^2 + \dots, & |x - x_0| \ll 1, \quad x_0 = 0.924, \\ -\frac{1}{2x^2} \left(1 + \frac{3}{2x^2} + \dots \right), & |x| \gg 1. \end{cases} \quad (20)$$

Generally speaking, a complete solution of (19), $v_0 = v_0(\theta, u_0)$, can only be found by numerical means. However, for the two special cases of interest

- i) no ion current in the wave frame, *i.e.* $u_0 \approx 0$,
- ii) vanishing total current, *i.e.* $u_0 = r^{-1}v_0$,

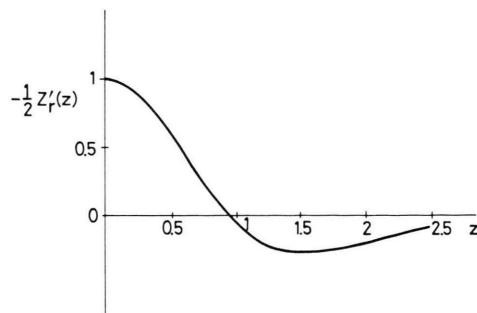


Fig. 2. Minus one half of the real part of the derivative of the plasma dispersion function for real arguments.

one can exploit (19) analytically. Since the electron and ion current, respectively, are given [12] by

$$j_e = -v_0, \quad j_i = r\tilde{A}u_0 \approx ru_0, \quad (21)$$

where $r \equiv \left(\frac{m_e T_i}{m_i T_e} \right)^{1/2}$, the first case implies, $u_0 \approx 0$, and the second one, $u_0 = r^{-1}v_0$, as already noted.

For $u_0 \equiv 0$, (19) becomes

$$\theta + Z_r'(v_0/\sqrt{2}) \approx 0$$

which means in view of $\theta > 0$ that $v_0/\sqrt{2}$ has to lie in the range where $-\frac{1}{2} Z_r'(v_0/\sqrt{2})$ is negative, *i.e.* $v_0/\sqrt{2} \gtrsim 0.924$. The electrons have to show a finite drift, and the plasma is carrying a free total current. Furthermore, since $-\frac{1}{2} Z_r'(x)$ has an absolute minimum of -0.285 we get the inequality $\theta < 0.285$. With other words, in the case of no ion drift – the DL is resting in the ion frame – the temperature ratio has to satisfy

$$T_i/T_e > 3.5 \quad (22)$$

which is just the opposite range required for ion acoustic waves (see also next section). If $u_0 = 0$ is released and the ions acquire a small drift, the situation becomes more favourable because $-\frac{1}{2} Z_r'(u_0/\sqrt{2})$ decreases and, therefore, θ from (19) can be larger than 0.285 or T_i/T_e smaller than 3.5.

In the second case, $u_0 = r^{-1}v_0$, which is assumed to be large, we can make use of (20) and get

$$-(m_e/m_i v_0^2) - \frac{1}{2} Z_r'(v_0/\sqrt{2}) \approx 0 \quad (23)$$

which again has the solution $v_0/\sqrt{2} \approx 0.924$. This case is formally achieved by setting $\theta = 0$, or, in view of (7), by assuming infinite ion mass (immobile ions), $n_i = \text{const} = 1$.

The latter assumption was made in deriving electron hole equilibria [9, 16, 20]. As already said, the discussion just presented applies for electron holes as well. With case i) we thus have obtained an extension of the electron hole theory to finite

plasma currents, at least in concern with the dispersion relation, with other words, if the electron hole is resting in the ion reference frame, the temperatures have to satisfy $T_i/T_e > 3.5$; on the other hand, if there is no plasma current at all, there is no such a restriction on T_i/T_e , and the old electron hole theory is valid.

With respect to DLs the situation is different:

DLs do require density variations for both species (especially if the densities are expected to be asymptotically different) and, hence, the case ii) does not apply.

Our preliminary conclusion, therefore, is that the main range for DLs of this kind is given by case i). Of course, this should be strengthened (or not) by a more detailed, completed analysis which includes a numerical evaluation of (12), (15) and (16).

We now prove the existence of solutions. Since (16) is rather complicated we consider two special cases.

$$\text{a) } |B_2| \ll |C_1| \psi^{1/2}.$$

This inequality applies for $\alpha \approx 1$, and $u_0 \approx 0$ namely, case i), the only case we have to consider here. In this case the first term in (15)–(18) can be dropped, and $V < 0$ in (16) or $V''(\Phi) < 0$ at $\Phi = 0$, and $\Phi = \psi$ implies $C_1 > 0$. The nonlinear dispersion relation, $A = \frac{2}{3} C_1 \psi$ (note that $C_2 \approx 0$) is solved in the limit $\theta \ll 1$ by

$$v_0 = 1.305 [1 + \theta - (2 C_1/3) \psi]. \quad (24)$$

Both expressions show that this kind of DL is based upon the slow electron acoustic mode [17, 19] (as the electron hole is), and this is why it is termed after that.

The electrostatic potential, obtained by solving (12) with V given by

$$-V(\Phi) = \frac{1}{3} C_1 \Phi^2 [\psi^{1/2} - \Phi^{1/2}]^2 \quad (25)$$

then reads [18]:

$$\Phi(x) = \frac{\psi}{4} [1 + \tanh \kappa x]^2, \quad \kappa = \left(\frac{C_1 \psi}{24} \right)^{1/2}. \quad (26)$$

The first condition in (15), $B_1 \approx -\frac{5}{3} C_1 \psi^{1/2}$, together with $C_1 > 0$, and with v_0 given by (24) or (25), respectively, implies

$$\beta < 0.71. \quad (27)$$

β has to be sufficiently negative, corresponding to a trapped electron region which is excavated. It's not surprising that the same condition holds for electron holes, too, where the electron distribution has to be of vortex-type [9, 16, 20]. Here, the electron phase space configuration resembles a tuning fork, as indicated in Figure 1b.

We further note that the density which is unity at $\psi = 0$, decreases with increasing Φ , as seen from (7), for example, in the limit $u_0 = 0$ when n_i becomes $n_i(\Phi) \approx 1 - \theta \Phi + \dots$. Hence, a potential increase is accompanied by a density decrease, as indicated in Figure 1b.

$$\text{b) } |B_2| \gg |C_1| \psi^{1/2}.$$

Now we drop the second term in (15)–(18). The necessary condition becomes $B_2 < 0$. The nonlinear dispersion relation can be solved as before, and we get (case i))

$$v_0 = 1.305 (1 + \theta + 2 B_2 \sqrt{\psi}), \quad (28)$$

where $\theta \ll 1$ is assumed. $B_2 = B_1 < 0$ from (15) implies that $\beta < -0.71$, and that $\alpha < 1$, indicating that the same trapped particle configurations are involved. The inequality b) is satisfied if β is sufficiently negative and $\theta \sim 0$ ($\psi^{1/4}$).

In case b), however, $\Phi(x)$ cannot be expressed analytically and can be obtained by a numerical treatment of (12) only.

If neither a) nor b) hold the general expressions, (15) and (16) have to be used, and the solution is gain obtainable by numerical means only.

In summary, this type of DL is based on the slow electron acoustic mode and requires a tuning fork-type electron phase space configuration. The restrictions on the ions are less severe as far as the drift and the trapping parameter are involved. The ion temperature, however, has to be sufficiently hot.

3. The Slow Ion Acoustic Double Layer (SIADL)

The analysis for the second type of weak monotonic transitions is facilitated by shifting the potential $\Phi \rightarrow \Phi - \psi$. The reason for this shift is, that it is then again the limit $\Phi \rightarrow 0$ for which the trapped particle component of the relevant species (this time the ion species) vanishes; $\Phi = 0$ is the maximum, and it holds $-\psi \leq \Phi \leq 0$, as indicated in Figure 1c.

The distributions are accordingly given by

$$\begin{aligned} f_e(x, v) &= \hat{A} (2\pi)^{-1/2} \begin{cases} \exp\{-\frac{1}{2}[\sigma_e(v^2 - 2(\Phi + \psi))^{1/2} - v_0]^2\}, & \varepsilon_e > 0, \\ \exp(-v_0^2/2) \exp[-\beta(v^2 - 2(\Phi + \psi))/2], & \varepsilon_e \leq 0, \end{cases} \\ f_i(x, u) &= (2\pi)^{-1/2} \begin{cases} \exp\{-\frac{1}{2}[\sigma_i(u^2 + 2\theta\Phi)^{1/2} - u_0]^2\}, & \varepsilon_i > 0, \\ \exp(-u_0^2/2) \exp[-\alpha(u^2 + 2\theta\Phi)/2], & \varepsilon_i \leq 0, \end{cases} \end{aligned} \quad (29)$$

where σ_e , σ_i and θ are defined as before, and ε_e and ε_i , respectively, are redefined by

$$\varepsilon_e \equiv [v^2 - 2(\Phi + \psi)]/2, \quad \varepsilon_i \equiv (u^2 + 2\theta\Phi)/2. \quad (30)$$

The densities become

$$\begin{aligned} n_e(\Phi) &= \hat{A} \exp(-v_0^2/2) [F(v_0^2/2, \Phi + \psi) \\ &\quad + T_{\pm}(\beta, \Phi + \psi)], \\ n_i(\Phi) &= \exp(-u_0^2/2) [F(u_0^2/2, -\theta\Phi) \\ &\quad + T_{\pm}(\alpha, -\theta\Phi)] \end{aligned} \quad (31)$$

which by using (4) and (5), and by applying the normalization $n_e = n_i = 1$ at $\Phi = 0$, reduce in the small amplitude limit to

$$\begin{aligned} n_e(\Phi) &= 1 - \frac{1}{2} Z'_r(v_0/\sqrt{2}) \Phi \\ &\quad + \frac{4}{3} b(\beta, v_0) [\psi^{3/2} - (\Phi + \psi)^{3/2}] \\ &\quad + \frac{\Phi}{2} \{G(v_0)(\Phi + 2\psi) \\ &\quad - \frac{1}{2} [Z'_r(v_0/\sqrt{2})]^2 \psi\} + \dots, \\ n_i(\Phi) &= 1 + \frac{\theta}{2} Z'_r(u_0/2) \Phi - \frac{4}{3} b(\alpha, u_0) (-\theta\Phi)^{3/2} \\ &\quad + \frac{1}{2} G(u_0) (\theta\Phi)^2 + \dots \end{aligned} \quad (32)$$

Substituting these expressions into

$$\Phi'' = n_e - n_i = -V'(\Phi)$$

we get

$$\begin{aligned} -V'(\Phi) &= A\Phi - B_1[\psi^{3/2} - (\psi + \Phi)^{3/2}] \\ &\quad - B_2(-\Phi)^{3/2} + C_1\Phi^2 + C_3\Phi\psi, \end{aligned} \quad (33)$$

where A , B_1 , B_2 , and C_1 are defined as before, (10), and C_3 is given by

$$C_3 \equiv G(v_0) - [\frac{1}{2} Z'_r(v_0/\sqrt{2})]^2. \quad (34)$$

The boundary condition, $V'(-\psi) = 0$, gives

$$0 = A + (B_1 + B_2) \psi^{1/2} - C_1 \psi + C_3 \psi, \quad (35)$$

and the integration of (33) results in

$$\begin{aligned} -V(\Phi) &= \frac{A}{2} \Phi^2 - B_1 \{\Phi \psi^{3/2} + \frac{2}{5} [\psi^{5/2} - (\psi + \Phi)^{5/2}]\} \\ &\quad + \frac{2}{5} B_2 (-\Phi)^{5/2} + \frac{C_1}{3} \Phi^3 + \frac{C_3 \Phi^2}{2} \psi, \end{aligned} \quad (36)$$

where again $V(0) = 0$ is assumed. The nonlinear dispersion relation, $V(-\psi) = 0$, reads

$$0 = A + \frac{2}{5} [2B_2 + 3B_1] \psi^{1/2} - \frac{2}{3} C_1 \psi + C_3 \psi. \quad (37)$$

A comparison of (35) and (37) with (11) and (14) shows that simply B_1 and B_2 are interchanged, that C_1 changes the sign, and that C_2 is replaced by C_3 . The discussion is, therefore, completely analogous to the previous section and can be shortened appreciably. Expressing A and B_2 by B_1 , C_1 and C_3 , we obtain

$$\begin{aligned} B_2 &= B_1 + \frac{5}{3} C_1 \psi^{1/2}, \\ A &= -2B_1 \psi^{1/2} - (\frac{2}{3} C_1 + C_3) \psi. \end{aligned} \quad (38)$$

Note that the first equations in (15) and (38) are equal. By insertion of these quantities into (36) we find

$$\begin{aligned} -V(\Phi) &= B_1 \{-\Phi \psi^{1/2} (\psi + \Phi) \\ &\quad - \frac{2}{5} [\psi^{5/2} - (-\Phi)^{5/2} - (\psi + \Phi)^{5/2}]\} \\ &\quad - \frac{C_1}{3} \Phi^2 [\sqrt{\psi} - \sqrt{-\Phi}]^2 \end{aligned} \quad (39)$$

(39) equals (16) if the replacement $(B_1, C_1, \Phi) \rightarrow (B_2, -C_1, -\Phi)$ is made. The interchange of B_1 and B_2 means that the species are interchanged. This implies that this kind of DL is subject to the following inequalities

$$T_e/T_i > 3.5, \quad \alpha < 0.71, \quad (40)$$

noting that only the case $v_0 \approx 0$ is left over because case i) $v_0 = 0$ (electron rest frame) and case ii) $v_0 = r u_0 \approx 0$ (no total current) merge for this DL.

The electrons must be sufficiently hot, and the trapped ion region has to be excavated as it holds for ion holes [9, 16, 21]. This type of DL is, therefore, a descendant of ion holes.

If the second term in (39) dominates, the solution is given by [18]

$$\Phi(x) = \frac{-\psi}{4} [1 + \tanh \kappa x]^2, \quad \kappa = \left(\frac{-C_1 \psi}{24} \right)^{1/2},$$

$$u_0 = 1.305 \left[1 + \theta^{-1} + \frac{2C_1}{3\theta} \psi \right], \quad (41)$$

where the latter assumes $\theta \gg 1$ and represents the nonlinear version of the slow ion acoustic mode [17, 19, 21].

In the limit $\Phi \rightarrow -\psi$ we get from (32)

$$n_e(-\psi) \cong 1 - \psi + \dots, \quad (42)$$

which shows that the density decreases with decreasing Φ . Hence, this DL is correctly illustrated in Figure 1c.

To sum up, this type of DL is based on the slow ion acoustic mode, and the ions necessarily have to acquire a tuning-fork configuration. It exists for sufficiently hot electrons like ion acoustic modes.

4. Stability of Weak Double Layers

In this section we are concerned with a linear stability analysis of the DL equilibria derived in the previous sections. Due to the lack of an energy principle (see below), a normal mode analysis is invoked. We shall first discuss the usual perturbation approach for longitudinal modes in the asymptotic region, and then perform a two-dimensional analysis with respect to localized electrostatic modes.

a) Asymptotic stability analysis

DLs do have the unique feature of being nonlinear and of providing a non-trivial homogeneous plasma state in both asymptotic regions $|x| \rightarrow \infty$. This means that in these regions standard perturbation theory can be applied [14]. The discussion is further simplified if we restrict the analysis to that asymptotic region of each DL in which a two-stream situation prevails namely $x \rightarrow -\infty$ for SEADL, and $x \rightarrow +\infty$ for SIADL. In this region the two prongs of the tuning-fork distribution are

united and form together with the second species a double hump, Maxwellian situation. This means that Stringer's results can be applied directly to these regions [22]. For SEADL we get in the low potential region, assuming the typical parameters $u_0 = 0$, $\alpha = 1$, $T_i/T_e \gtrsim 10$, and using Fig. 2 of reference [22], a critical drift velocity $v_0^* \gtrsim 3$ for the onset of instability. The actual value of v_0 , given by (24), however, is $v_0 \lesssim 1.44$ which implies stability. Similarly, for SIADL in the high potential region, assuming $v_0 = 0$, $\beta = 1$, $T_e/T_i \approx 10$ (100), we get $u_0^* = 6.8$ (3.4) as the critical drift velocity, whereas u_0 is actually given by (41) and becomes $u_0 \approx 1.44$ (1.32), which again implies stability.

For the remaining two regions the distributions are multiple-humped, and the analysis becomes more intricate asking, in generality, for a Nyquist technique. We haven't performed such kind of analysis as we don't expect drastic changes in the stability behaviour by going from the one to the other asymptotic region. This expectation is supported by the next subsection in which the DL's stability with respect to one-dimensional electrostatic perturbations is proven within certain limits.

On the other hand, numerical and experimental observations [23–31], where usually higher amplitudes of the DL's are involved, do show non-negligible wave activities in both asymptotic regions, indicating instability there. We, therefore, expect a change of the stability behaviour for finite amplitudes $\psi \gtrsim 0$ (1). In this regime where other stability theories fail, including the one we'll present now, the asymptotic normal mode analysis will prove to be a useful method as it is by far the simplest one.

On the other hand, experiments show that the noise is peaked in the DL region, *i.e.* in the region where the transition occurs, and, furthermore, the modes are found to propagate with a finite angle with respect to the x -direction. Both suggest a two-dimensional analysis based on localized modes.

b) Two-dimensional analysis of localized electrostatic modes

We want to solve the two-dimensional Vlasov-Poisson system

$$[\partial_t + \mathbf{v} \cdot \nabla + \nabla \Phi \cdot \partial_v] f_e = 0,$$

$$[\partial_t + \mathbf{u} \cdot \nabla - \theta \nabla \Phi \cdot \partial_u] f_i = 0, \quad (43)$$

$$\Delta\Phi = n_e - n_i, \quad (44)$$

by making the following ansatz

$$\begin{aligned} f_e &= F_{0e}(x, v_x, v_y), \quad + f_{1e}(x, y, v_x, v_y, t), \\ f_i &= F_{0i}(x, u_x, u_y), \quad + f_{1i}(x, y, u_x, u_y, t), \\ \Phi &= \Phi_0(x) + \Phi_1(x, y, t). \end{aligned} \quad (45)$$

Here, subscript 0 denotes the equilibrium quantities, i.e. $\Phi_0(x)$ is the DL potential of the previous sections and

$$F_{0e} = f_{0e}(x, v_x) (2\pi)^{-1/2} \exp(-\varepsilon_\perp) \quad (46)$$

is the electron equilibrium distribution function extended into the two-dimensional velocity space, $\varepsilon_\perp = v_y^2/2$ representing the perpendicular unperturbed energy. A similar expression holds for the ions, and $v(u)$ is now denoted by $v_x(u_x)$, respectively.

It is worth noticing that this ansatz does not fit into the limited class of distributions which are isotropic in velocity space, i.e. which depend on the unperturbed two-dimensional Hamiltonian

$$H_0 = \frac{1}{2}(v_x^2 + v_y^2) - \Phi_0(x) \equiv \varepsilon_\parallel + \varepsilon_\perp$$

for electrons, and

$$H_0 = \frac{1}{2}(u_x^2 + u_y^2) + \theta \Phi_0(x) \equiv \varepsilon_\parallel + \varepsilon_\perp$$

for ions, respectively,

where for simplicity the subscript j in $\varepsilon_{\parallel j}(\varepsilon_{\perp j})$, $j = i, e$ has been dropped.

Variational principles [32–34], for example, being based on the assumption of isotropy, would not apply here. In fact, one can additionally show that this restricted class of distributions does not allow any bounded electrostatic solution at all, so that this class of equilibrium solutions is empty indeed.

The proof is simple and can be written in two lines: With such restricted distributions, Poisson's equation becomes

$$\begin{aligned} \Phi_0''(x) &= 2\pi \left[\int_{-\Phi_0}^{\infty} dH_0 f_{0e}(H_0) - \int_{\theta\Phi_0}^{\infty} dH_0 f_{0i}(H_0) \right] \\ &\equiv -V''(\Phi_0), \end{aligned} \quad (47)$$

from which

$$-V'''(\Phi_0) = 2\pi [f_{0e}(-\Phi_0) + \theta f_{0i}(\theta\Phi_0)] \geq 0, \quad (48)$$

follows. The classical potential has a non-positive curvature everywhere, and consequently $\Phi_0(x)$ cannot be bounded.

Nontrivial, bounded one-dimensional, electrostatic solutions can, hence, only be obtained by a broader class of distributions like the one we have used here.

A Fourier ansatz is now made for perturbations in the homogeneous y and t directions, e.g. $\Phi_1(x, y, t) \sim \exp\{i(ky - \omega t)\}$, where k denotes the perpendicular wave number. $\text{Im } \omega > 0$ represents growth of the perturbations.

Linearizing (43) with respect to the perturbed quantities we obtain

$$[-i\hat{\omega}_e + L_e]f_{1e} = -[\partial_{\varepsilon_\parallel} F_{0e} L_e + i k v_y \partial_{\varepsilon_\perp} F_{0e}] \Phi_1, \quad (49)$$

$$[-i\hat{\omega}_i + L_i]f_{1i} = +[\partial_{\varepsilon_\parallel} F_{0i} L_i + i k u_y \partial_{\varepsilon_\perp} F_{0i}] \theta \Phi_1,$$

where

$$\begin{aligned} \hat{\omega}_e &\equiv \omega - k v_y, \\ \hat{\omega}_i &\equiv \omega - k u_y, \end{aligned} \quad (50)$$

represents the doppler-shifted frequencies and

$$\begin{aligned} L_e &\equiv v_x \partial_x + \Phi_0'(x) \partial_{v_x}, \\ L_i &\equiv u_x \partial_x - \theta \Phi_0'(x) \partial_{u_x}, \end{aligned} \quad (51)$$

are the unperturbed Vlasov- or Liouville-operators.

The transformation [35, 36]

$$\begin{aligned} g_e &= f_{1e} + \partial_{\varepsilon_\parallel} F_{0e} \Phi_1, \\ g_i &= f_{1i} - \partial_{\varepsilon_\parallel} F_{0i} \theta \Phi_1, \end{aligned} \quad (52)$$

casts the system (49) into

$$\begin{aligned} [-i\hat{\omega}_e + L_e] g_e &= -i[\hat{\omega}_e \partial_{\varepsilon_\parallel} + k v_y \partial_{\varepsilon_\perp}] F_{0e} \Phi_1, \\ [-i\hat{\omega}_i + L_i] g_i &= +i[\hat{\omega}_i \partial_{\varepsilon_\parallel} + k u_y \partial_{\varepsilon_\perp}] F_{0i} \theta \Phi_1, \end{aligned} \quad (53)$$

and Poisson's equation becomes

$$\mathcal{A} \Phi_1 = \int g_e d^2v - \int g_i d^2u, \quad (54)$$

where the field operator \mathcal{A} [36] is defined by

$$\begin{aligned} \mathcal{A} &\equiv \partial_x^2 - k^2 + \int d^2v \partial_{\varepsilon_\parallel} F_{0e} + \theta \int d^2u \partial_{\varepsilon_\parallel} F_{0i}, \\ &= \partial_x^2 - k^2 - n'_{0e}(\Phi_0) + n'_{0i}(\Phi_0), \\ &= \partial_x^2 - k^2 + V'''(\Phi_0). \end{aligned} \quad (55)$$

Prime means differentiation with respect to Φ_0 , and $V(\Phi_0)$ is again the classical potential.

The linear system (53) is solvable by the method of characteristics and a partial integration technique

assuming no contributions at infinity, $|\Phi_1| \rightarrow 0$ for $|x| \rightarrow \infty$, can be used to write down the solutions [37] as

$$\begin{aligned} g_e &= \left[\partial_{\varepsilon_{\parallel}} + \frac{k v_y}{\hat{\omega}_e} \partial_{\varepsilon_{\perp}} \right] F_{0e} \left\{ 1 - \frac{i}{\hat{\omega}_e} v_x \partial_x \left[1 - \frac{i}{\hat{\omega}_e} v_x \partial_x (1 - \dots) \right] \right\} \Phi_1, \\ g_i &= - \left[\partial_{\varepsilon_{\parallel}} + \frac{k u_y}{\hat{\omega}_i} \partial_{\varepsilon_{\perp}} \right] F_{0i} \left\{ 1 - \frac{i}{\hat{\omega}_i} u_x \partial_x \left[1 - \frac{i}{\hat{\omega}_i} u_x \partial_x (1 - \dots) \right] \right\} \Phi_1. \end{aligned} \quad (56)$$

(56) can also be obtained directly by Taylor expanding the formal solutions of (53). Substituting g_e and g_i into (54) we get [38–40]

$$\begin{aligned} &\left\{ \Lambda(x) - \int d^2v \left[\partial_{\varepsilon_{\parallel}} + \frac{k v_y}{\hat{\omega}_e} \partial_{\varepsilon_{\perp}} \right] F_{0e} \left[1 - \frac{i v_x}{\hat{\omega}_e} \partial_x - \frac{1}{\hat{\omega}_e^2} v_x^2 \partial_x^2 + \dots \right] \right. \\ &\quad \left. - \int d^2u \left[\partial_{\varepsilon_{\parallel}} + \frac{k u_y}{\hat{\omega}_i} \partial_{\varepsilon_{\perp}} \right] F_{0i} \left[1 - \frac{i u_x}{\hat{\omega}_i} \partial_x - \frac{1}{\hat{\omega}_i^2} u_x^2 \partial_x^2 + \dots \right] \theta \right\} \Phi_1 = 0, \end{aligned} \quad (57)$$

which is abbreviated by

$$K(x, \omega, k) \Phi_1(x) = 0. \quad (58)$$

Equation (58) represents together with appropriate boundary conditions an inhomogeneous eigenvalue problem with eigenvalue ω and eigenfunction (normal mode) $\Phi_1(x)$ for a given k .

In a forthcoming paper [41] it will be shown that the infinite order differential operator $K(x, \omega, k)$ is self-adjoint for the class of small amplitude, localized Vlasov equilibria. A similar proof, restricted to a standing wave pattern where no particle drifts are involved, has been given by [40].

Self-adjointness implies that the error in the eigenvalue $\delta\omega = \omega - \tilde{\omega}$ is second order in δ , where δ is a smallness quantity, $\delta \ll 1$, provided that the difference between an exact normal mode Φ_1 and an approximate solution $\tilde{\Phi}_1$ can be characterized by δ [40]. Here $\tilde{\omega}$ and the test function $\tilde{\Phi}_1$ approximately satisfy (58): $K(x, \tilde{\omega}, k) \tilde{\Phi}_1 \approx 0$.

The test function we are choosing is an eigensolution of the field operator Λ being defined by the following Schrödinger eigenvalue problem [34, 35],

$$\Lambda \eta_n = -(\lambda_n + k^2) \eta_n. \quad (59)$$

Making use of the explicit form of $\Phi_0(x)$ given by (26), and of $V''(\Phi_0)$ given by (18) and analogously for SIADL, the potential $U(x)$ in the Schrödinger operator Λ becomes

$$U(x) \equiv -V''(\Phi_0) \quad (60)$$

$$= \frac{C_1 \psi}{2} \left\{ [\tanh(\kappa x) - \frac{1}{4}]^2 - \frac{11}{48} \right\} \equiv C_1 \psi u(\xi),$$

where $\xi = \kappa x$. The function $u(\xi)$ given by one half of the curled bracket in (60), is shown in Figure 3. It resembles a soft-core nuclear potential and possesses one bound-state given by $\lambda_0 = 0$ (indicated by the heavily drawn line in Figure 3).

The corresponding eigenfunction $\eta_0(x)$ becomes

$$\eta_0(x) \sim \cosh^{-2}(\kappa x) [\tanh(\kappa x) + 1]. \quad (61)$$

It is shown to be proportional to $\Phi'_0(x)$ and, therefore, represents the translational mode of system being obtained by an infinitesimal uniform displacement of the original equilibrium which is recovered in (49)–(53) by setting $k = \omega = g_e = g_i = 0$.

Next we replace Φ_1 in (57) by $\tilde{\Phi}_1 \equiv \eta_0(x)$ and perform the scalar product in the space of square-integrable functions by multiplication of (57) with $\eta_0(x)$ and integration over x .

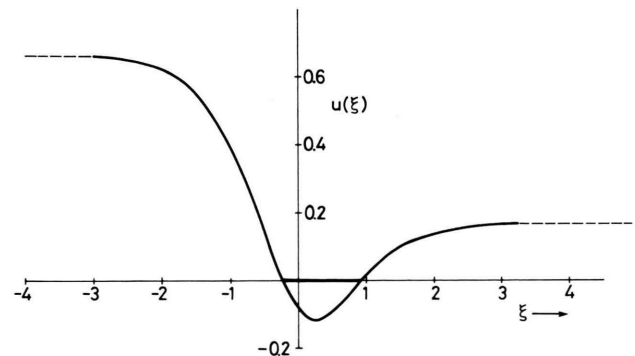


Fig. 3. The normalized Schrödinger-potential $u(\xi)$ as a function of $\xi = \kappa x$. It possesses one bound state at zero energy.

The result represents the dispersion relation or, more precisely, an approximation to that, and can be written [41] as

$$k^2 c_0 + A_1 + A_2 + \sum_{n=1}^{\infty} (-1)^n c_{2n} \{I_{\parallel}^{2n}(\zeta, v_0) + I_{\perp}^{2n}(\zeta, v_0) + \theta [I_{\parallel}^{2n}(\zeta, u_0) + I_{\perp}^{2n}(\zeta, u_0)]\} = 0 \quad (62)$$

with the following definitions

$$\begin{aligned} c_n &\equiv \int_{-\infty}^{+\infty} dx \eta_0(x) \frac{d^n}{dx^n} \eta_0(x) = \int_{-\infty}^{+\infty} dx \eta_0(x) \eta_0^{(n)}(x), \\ A_1 &\equiv \int_{-\infty}^{+\infty} dx \eta_0(x) \left[\int d^2v \partial_{\varepsilon_{\parallel}} F_{0e} \right. \\ &\quad \left. + \theta \int d^2u \partial_{\varepsilon_{\parallel}} F_{0i} \right] \eta_0(x) \\ &= \int_{-\infty}^{+\infty} dx \eta_0(x) V''(\Phi_0(x)) \eta_0(x), \\ A_2 &\equiv \int_{-\infty}^{+\infty} dx \eta_0(x) \left[\int d^2v \frac{k v_y}{\hat{\omega}_e} \partial_{\varepsilon_{\perp}} F_{0e} \right. \\ &\quad \left. + \theta \int d^2u \frac{k u_y}{\hat{\omega}_i} \partial_{\varepsilon_{\perp}} F_{0i} \right] \eta_0(x) \\ &= -\frac{(1+\theta)}{2} Z'(\zeta) c_0 + O(\psi), \end{aligned} \quad (63)$$

$$\begin{aligned} I_{\parallel}^n(\zeta, v_0) &\equiv \int d^2v \frac{v_x^n}{\hat{\omega}_e^n} \partial_{\varepsilon_{\parallel}} F_{0e} + O(\psi) \\ &= -(n-1) M_{n-2}^{(0)}(v_0) \Omega_n(\zeta) + O(\psi), \end{aligned}$$

$$M_0^{(0)} = 1, \quad M_2^{(0)} = 1 + w^2, \quad M_4^{(0)} = 3 + 6w^2 + w^4,$$

where $\zeta \equiv \frac{\omega}{\sqrt{2}k}$, $Z(\zeta)$ is the plasma dispersion function [20] and

$$(m; d; v) \equiv m(m+d)(m+2d) \dots (m+(v-1)d)^*.$$

$I_{\parallel}^n(\zeta, u_0)$ and $I_{\perp}^n(\zeta, u_0)$ stemming from the ions are defined analogously. The first terms in (63), using (61) and (25), read

$$c_0 = \frac{8}{5K}, \quad c_2 = -\frac{64}{35}K; \quad A_1 = \frac{48 \cdot 4}{35 \cdot 3}K, \quad (64)$$

where

$$K = \sqrt{c_1 \psi / 24}.$$

Retaining the $n=1$ term in the series of (62) only, we get as a first approximation to the dispersion relation

$$\begin{aligned} &k^2(k^2 + 1 + a + \theta) - a(v_0^2 + u_0^2) \\ &= \zeta Z(\zeta) \left\{ \frac{a}{2} [(1+\theta) + 3(v_0^2 + \theta u_0^2)] - k^2(1+\theta) - a\zeta^2 [1 + v_0^2 + \theta(1 + u_0^2)] \right\} \\ &\quad - a\zeta^2 [1 + v_0^2 + \theta(1 + u_0^2)], \end{aligned} \quad (65)$$

where $a \equiv \frac{8}{7} K^2 = c_1 \psi / 21$, and use has been made of $Z'(\zeta) = -2[1 + \zeta Z(\zeta)]$.

$$\begin{aligned} I_{\perp}^n(\zeta, v_0) &\equiv \int d^2v \frac{k v_y v_x^n}{\hat{\omega}_e^{n+1}} \partial_{\varepsilon_{\perp}} F_{0e} + O(\psi) \\ &= M_n^{(0)}(v_0) \left(1 + \frac{\zeta}{n} \frac{d}{d\zeta} \right) \Omega_n(\zeta) + O(\psi), \end{aligned}$$

$$M_n \equiv \int dv_x v_x^n f_0(v_x) = M_n^{(0)} + O(\psi),$$

$$\begin{aligned} \Omega_n &\equiv (2\pi)^{-1/2} \int dv_y \frac{\exp(-v_y^2/2)}{\hat{\omega}_e^n} \\ &= \frac{(-1)^n}{(n-1)!} (\sqrt{2}k)^{-n} Z^{(n-1)}(\zeta), \quad n \geq 2, \end{aligned}$$

$$M_n^{(0)}(w) = \sum_{v=0}^n \left(\frac{n}{2v} \right) w^{2v} \left(1; 2; \frac{n}{2} - v \right),$$

In the limit $u_0=0$, $\theta \ll 1$ ($v_0=1.305$), one recovers the same expression than the one derived for electron holes [37]:

$$\begin{aligned} &k^2(k^2 + 1 + a) - a v_0^2 \\ &= \zeta Z(\zeta) \left[\frac{a}{2} (1 + 3v_0^2) - k^2 - a\zeta^2 (1 + v_0^2) \right] \\ &\quad - a\zeta^2 (1 + v_0^2). \end{aligned} \quad (66)$$

* Gröbner-Hofreiter, *Integraltafel*, Springer-Verlag, Wien 1961.

Its evaluation shows that there exists an unstable branch of purely growing modes:

$$\operatorname{Re} \omega = 0, \quad \operatorname{Im} \omega > 0 \quad \text{for} \quad 0 < k \leq k_c \equiv a^{1/2} v_0.$$

These are two-dimensional modes whose typical transverse wavelength is comparable with the scale length of the inhomogeneity. The growth rate γ , being normalized by ω_{pe} , scales with $\psi^{1/4}$, the one-fourth power of the potential jump. Hence, the stronger the DL the faster the instability will arise. There is no instability in one dimension, $k = 0$.

Our conclusion, therefore, is that within the assumed approximations weak DL's are transversal unstable with respect to electrostatic modes that are local in x . It is, however, unclear yet how these results will be modified by taking into account more, or better all terms in the series of (62), a question which needs further investigations.

Nevertheless, formula (62) provides the first well-founded approximation to a (unknown) dispersion relation of linearized modes in the presence of weak DL equilibria.

5. Comparisons with Experiments – Conclusions

We are going to terminate this paper with a brief discussion. The question we want to discuss is whether or not these theoretical results can be applied experimentally.

For this reason we first summarize the main characteristic properties of the two new DLs by pointing out especially their range of validity.

First of all, the assumption of small amplitudes was made. Although it would be very surprising if these two new classes could not be extended into the finite amplitude regime, we should not forget this limitation. It is possible that with increasing ψ the properties experience modifications which are not foreseeable. Nevertheless, by extrapolating the results into the finite amplitude regime, where the experiments are being placed, we are able to make comparisons.

These comparisons are facilitated by the fact that several properties can be formulated already on the macroscopic level, as there are:

- the drop (SEADL) or increase (SIADL) of the density with increasing potential,
- the requirement of non-isothermal conditions: $T_i > T_e$ (SEADL) and $T_e > T_i$ (SIADL),

- enhanced noise in the DL region with finite k_\perp ,
- the need of a current for SEADL ($v_0 \approx 1 \rightarrow v_{De} \approx v_{the}$), and of almost current-free conditions for SIADL ($u_0 \approx 1 \rightarrow v_{Di} \approx v_{thi}$),
- neither the Bohm- nor the Langmuir condition have to be satisfied for both DLs.

Microscopically, the most salient features are:

- the tuning-fork-like phase-space configuration of electrons (SEADL) and of ions (SIADL),
- a strong component of trapped ions (SEADL) and of trapped electrons (SIADL), consistent with
- Maxwellian-like ion distributions for SEADL and Maxwellian-like electron distributions for SIADL in the entire region.

A look into the experimental (and numerical) literature shows that the vast majority of measured DLs neither belong to the class of SEADLs nor to the class of SIADLs. They usually satisfy Bohm-criteria, have drifting particles well separated from trapped particles in each species, join equal dense asymptotic plasmas, seem to be rather insensitive to the temperature ratio, and are found under strong current conditions, all attributes of the strong beam-type DL (see [21] for a further discussion).

There seem to be only two exceptions pointing at the experimental evidence of the extended class of weak double layers, a rather vague one for the SEADL and stronger one for the SIADL.

The double layer found in magnetic reconnection experiments [29, 42] carries some characteristic labels of the SEADL: It establishes only during current disruption indicating reduced current conditions. The density is greatly (~ 50 percent) reduced on the high potential side. The electron distribution function exhibits two humps on the high potential side, which are only slightly separated, whereas a second hump seems to be absent for ions on the low potential side. Plasma wave emission generated by beam-plasma instabilities may account for the partial relaxation of the electron distribution in comparison with the tuning-fork configuration as required from the theory of SEADLs.

More convincing is the second example. Hollenstein [30] investigated in detail turbulent potential structures in a long triple plasma device. By application of a pulsed voltage to the biasing grid he was able to record in space and time the evolution and subsequent formation of one or more

double layers. The temperature ratio in this experiment was $\theta = T_e/T_i = 10 > 3.5$, and the electron drift v_d was found to be as low as $v_d \lesssim 0.2 v_{the}$, so that the Bohm criterion was certainly violated. A remarkable depression of the density on the low potential side was observed. The turbulence associated with and induced by the potential structure was found to be dominated by low frequency turbulence ($\omega < \omega_{pi}/4$), having a propagating angle with respect to the electron drift of 90° . It was located in the structure on the low potential side. There is, hence, a striking similarity to our analytical results according to which nearly zero frequency modes with finite k_\perp should be excited within the double layer structure. It is furthermore interesting to experience that although the fluctuation level, including the high frequency components, was relatively high, $10^{-3} \lesssim (\delta n/n)^2 \lesssim 0.1$, it did not suppress or destroy the dc-DL structure.

The distribution functions were obtained for both species. Maxwellian-shaped velocity distributions for the electrons were found in the direction of the drift. A plot of f_e in the $z-t$ ($\cong x-t$) space (Fig. 4 of [30]) shows at $t \gtrsim 100 \mu s$, and $z \approx 40$ cm where the jump established clearly an acceleration and subsequent cooling of the drifting electrons. The distribution is single humped everywhere. Theoretically, we would describe this situation by setting $v_0 > 0$ and $\beta < 1$. More spectacular, however,

is the ion distribution function. It is seen to have split into a two-humped distribution on the low, and exhibits a Maxwellian-type pattern on the high potential side. Further downstream ($\Phi \rightarrow 0$) the two humps flatten and merge, releasing obviously their energy to the enhanced noise. We are, thus, almost automatically led to interpret this structure in the sense of a tuning-fork configuration in the ion phase space and hence, the whole potential as belonging to the class of SIADLs.

Finally, in an earlier investigation [25] it was noticed that this potential structure only occurred in the low or almost zero current limit (case I in [25]). If a finite current was drawn through the plasma (case II) no potential jump did arise.

We may, therefore, conclude after all that Hollenstein's experiment is the first manifestation of slow ion acoustic double layers (SIADL). More generally, we may state that it sometimes makes sense to apply a laminar theory to cases which, at a first glance, appear to be turbulent. The super-imposed ordered motion and the large amount of energy associated with introduce and represent a certain degree of persistence in the nonlinear structure with respect to the destructive effect of fluctuations. The gross coherent features are maintained in such situations.

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